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LETTER TO THE EDITOR

A lower bound for the spectrum of N-particle Hamiltonians

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Abstract

We consider the spectrum of an N-particle Hamiltonian $H = \sum_i (1/2m_i)\Delta_{\vec{r}_i} + \sum_{i<j} V_{ij}(\vec{r} - \vec{r}_{ji})$ with translation-invariant pair interactions V_{ij}. H acts in $L^2(R^{3N})$. Letting σ denote the spectrum we obtain the lower bound $\inf \sigma(H) \geqslant \sum_{i<j} \inf \sigma(h'_{ij})$ where $h'_{ij} = -(1/2\mu'_{ij})\Delta_{\vec{r}} + V_{ij}(\vec{r})$, $1 \leqslant i < j \leqslant N$ is the single-particle relative coordinate Hamiltonian with reduced mass $\mu'_{ij} = (N-1)\mu_{ij}$, $\mu_{ij} = m_i m_j (m_i + m_j)^{-1}$ acting in $L^2(R^3)$. In particular, if $\sigma(h'_{ij}) \subset [0, \infty)$ (for example, weak pair interactions) for all i, j then H has no negative energy spectrum. For example, if each $V_{ij}(\vec{r})$ is in $L^{3/2}(R^3)$ and sufficiently small it is known that $\sigma(h'_{ij}) \subset [0, \infty)$.

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We consider the spectrum of an N-particle Hamiltonian $H = \sum_i T_i + \sum_{i<j} V_{ij}$, where $T_i = -(1/2m_i)\Delta_{\vec{r}_i}$ and V_{ij} is a real translation-invariant pair interaction acting in $L^2(R^{3N})$.

One of the fundamental differences between classical and quantum mechanics is that for certain classes of physically important pair potentials the spectrum of the associated self-adjoint Hamiltonian operator H is bounded from below, although the classical Hamiltonian phase space function is not lower bounded. Sufficient conditions on V_{ij} for self-adjointness and lower boundedness are well known and given in theorems X.16, 17 and 19 of [1] (see also [2]).

Considering the case when the pair potentials go to zero at infinity there is the question of the relationship between the absence of bound states in the two-body problem and the absence of a negative energy spectrum for the N-particle Hamiltonian. The form of the bounds in [1] do not exclude the negative energy spectrum. However, using heavy functional analytic techniques, sufficient conditions on the pair potentials which imply the absence of a negative are obtained in theorem X.III.27 of [3]. In this case the potentials are required to be weaker as N increases.

Here we give another form of a lower bound for H in terms of the sum over pairs of lower bounds for each pair Hamiltonian, but with each mass increased by a factor of $N - 1$.

Pair-potential conditions which imply bounds of the form, $H \geqslant cN, c < 0$, which in turn imply thermodynamic stability are given in [4]. The more difficult problem of a precise bound of the form $H \geqslant cN, c > 0$, for finite density boson systems is treated in [5].

Turning now to our bound we assume that each V_{ij} is relatively bounded in the operator or form sense with respect to $T = \sum_i T_i$ with relative bound <1 (see [1, 2]). It is known that the associated self-adjoint Hamiltonian operator is bounded from below. We give another form of the lower bound by partitioning the Hamiltonian writing

$$H = \sum_{i \neq j} \left[\frac{1}{2(N-1)}(T_i + T_j) + \tfrac{1}{2} V_{ij} \right]$$

$$= \sum_{i<j} \left[\left(\frac{1}{(N-1)} \right)(T_i + T_j) + V_{ij} \right] = \sum_{i<j} H'_{ij}$$

then, for the case of operator relative boundedness, and letting σ denote the spectrum we have

$$\inf \sigma(H) = \inf_{\psi \in D(T), |\psi|=1} (\psi, H\psi) \geqslant \sum_{i<j} \inf \sigma(H'_{ij}) = \sum_{i<j} \inf \sigma(h'_{ij}) \qquad (1)$$

where $D(T)$ is the domain of $T = \sum_i T_i$ and h'_{ij} is the single-particle relative coordinate Hamiltonian

$$h'_{ij} = -\frac{1}{2\mu'_{ij}} + V_{ij}(\vec{r})$$

with $\mu'_{ij} = (N-1)\mu_{ij}$, $\mu_{ij} = m_i m_j/(m_i + m_j)$ acting in $L^2(R^3)$. The last equality in equation (1) is obtained by noting that H'_{ij} is unitarily equivalent to the sum of $I \otimes h'_{ij}$ and the centre of mass Hamiltonian

$$h'^{c}_{ij} \otimes I = \frac{1}{2(N-1)(m_i + m_j)} \Delta_{\vec{r}_{cm}}$$

acting in $L^2(R^3) \otimes L^2(R^3)$. h'^{c}_{ij} has a spectrum $[0, \infty)$.

In closing we recall some lower bounds for $h = t + v$, $t = -(1/2\mu)\Delta_{\vec{r}}$, $v = v(\vec{r})$, h acting in $L^2(R^3)$ (see [3–6]). These conditions on v ensure that the Neumann series for

$$(h - z)^{-1} = (t - z)^{-1}[1 + v(t - z)^{-1}]^{-1}$$

or

$$(h - z)^{-1} = (t - z)^{-1} v^{1/2} [1 + v^{1/2}(t - z)^{-1} v^{1/2}]^{-1} v^{1/2}$$

converge for z real and sufficiently large negative, i.e. these zs are not in the spectrum of h.

If $|v(t-z)^{-1}| < 1$ or $|v^{1/2}(t-z)^{-1}v^{1/2}| < 1$ the Neumann series converge. For example, with $v = v_2 + v_\infty$, $v_2 \in L^2$, $v_\infty \in L^\infty$, and letting HS denote the Hilbert–Schmidt norm we have

$$\left| v(t-z)^{-1} \right| \leqslant \left| v_2(t-z)^{-1} \right|_{\mathrm{HS}} + |v_\infty|_\infty \left| (t-z)^{-1} \right| \leqslant |v_2|_2 \frac{2\mu}{\sqrt{8\pi (|z| 2\mu)^{1/2}}} + |v_\infty|_\infty \frac{1}{|z|} < 1$$

which is not good enough to exclude the negative energy spectrum. For this exclusion we use the Rollnik condition (see [3, 6])

$$\left| v^{1/2}(t-z)^{-1} v^{1/2} \right| \leqslant \left| v^{1/2}(t-z)^{-1} v^{1/2} \right|_{\mathrm{HS}} \leqslant \frac{2\mu}{4\pi} \left[\int \frac{|v(x)||v(y)|}{|x-y|} \, \mathrm{d}x \, \mathrm{d}y \right]^{1/2} < 1. \qquad (2)$$

The Rollnik class is larger than $L^{3/2}$ and it is known that

$$\left[\int \frac{|v(x)||v(y)|}{|x-y|^2}\,\mathrm{d}x\,\mathrm{d}y\right]^{1/2} \leqslant c|v|_{3/2}$$

where the best possible value of c is given in [7]. Our condition on v in equation (2) is weaker than the one in [3], where v is required to belong to $L^{3/2-\varepsilon} \cap L^{3/2+\varepsilon}$ for some $\varepsilon > 0$.

As a concrete example the spherically symmetrical pair potential

$$v(\vec{r}) = c_1|\vec{r}|^{-\alpha} + c_2(1+|\vec{r}|)^{-\beta}$$

with $0 < \alpha < 2$, $\beta > 2$ (c_1 and c_2 are constants) is in the class of potentials for which our spectral bounds hold and the right-hand side of equation (1) is finite. The negative energy spectrum is absent for $|c_1|$ and $|c_2|$ sufficiently small (depending on N), so the inequality of equation (2) is satisfied. The condition $\beta > 2$ for the absence of a negative energy spectrum cannot be improved much as it is known (see [3]) that $\tilde{H}$ (H with the centre of mass removed) has an infinite number of negative energy bound states if $\beta < 2$ and c_2 is negative.

External single-particle potentials $v_i(\vec{r})$ can be included with $H'_{ij} + [1/(N-1)](v_i(\vec{r}_i) + v_j(\vec{r}_i))$ replacing H'_{ij} and the inf of the spectrum of these two-body Hamiltonians is to be taken.

Of course these considerations do not give any information on the type of spectrum.

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